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Stochastic differential equations driven by a fractional Brownian motion

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Resum *(CAT)*
En aquest treball s'estudien les equacions diferencials estocàstiques (EDEs) dirigides per un moviment brownià fraccionari (fBm) amb paràmetre de Hurst $H > 1/2$. Es defineix la integral estocàstica respecte al fBm i es demostra l'existència i unicitat de solucions. També s'introdueix el càlcul de Malliavin en el context del fBm, i es prova que, amb condicions més fortes en els coeficients, la llei de la solució és absolutament contínua. Finalment, es donen fites d'estil gaussià per la densitat d'una família d'EDEs.

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Abstract

In the thesis, we study from several points of view the so-called stochastic differential equations driven by a fractional Brownian motion with Hurst index $H > 1/2$. These objects are differential equations of the form

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\, \mathrm{d}B_s^H + \int_0^t b(s, X_s)\, \mathrm{d}s, \quad t \in [0, T], \tag{1}$$

where B^H is a fractional Brownian motion with $H \in (1/2, 1)$, that is, a centered stochastic Gaussian process with covariance function

$$R_H(t, s) := E(B_t^H B_s^H) = \frac{t^{2H} + s^{2H} - |t - s|^{2H}}{2}.$$

Notice that, in particular, when $H = 1/2$ then B^H is a standard Brownian motion. The first topic covered in the thesis is giving sense to an equation like (1). The fact that B^H is not a semimartingale makes it impossible to define the integral with respect to B^H in a similar way as it is defined for the standard Brownian motion. In virtue of the results of Young in [4] and the further contributions of Zähle in [5], we are able to define the integral with respect to B^H in the generalized Stieltjes sense. Once the stochastic integral is well-defined, we follow closely the arguments of Nualart and Răşcanu in [2] to prove the existence and uniqueness of solutions to a general SDE of the form (1).

Once we know we can talk about the solution to equation (1), then we want to study such solution from a probabilistic point of view. Using the Malliavin calculus (that is, the stochastic calculus of variations) we get to prove that under stronger hypothesis on the coefficients σ and b, the law of X_t is absolutely continuous with respect to the Lebesgue measure, so for each $t \in [0, T]$ X_t has a density function $p_t(x)$. In order to prove this result, we use the concepts of Malliavin differentiability in the fractional Brownian motion framework and Fréchet differentiability and we use the same techniques as in Nualart and Saussereau in [3].

Finally, using more sophisticated Malliavin calculus techniques and the method of Nourdin–Viens in [1] we are able to proof that the solution X_t to an equation of the type

$$X_t = x_0 + \int_0^t \sigma_s \, dB_s^H + \int_0^t b(s, X_s) \, ds, \tag{2}$$

where σ is deterministic, σ and b satisfy the same hypothesis as for the existence of the density function $p_t(x)$ and we assume further that there exist $0 < \lambda < \Lambda$ such that $\lambda < \sigma_s < \Lambda$, then the density $p_t(x)$ for $t \in (0, T]$ is bounded in the following way:

$$\frac{E(|X_t - m_t|)}{2\Lambda^2 t^{2H}} \exp\left(-\frac{(x - m_t)^2}{2\lambda^2 t^{2H}}\right) \leq p_t(x) \leq \frac{E(|X_t - m_t|)}{2\lambda^2 t^{2H}} \exp\left(-\frac{(x - m_t)^2}{2\Lambda^2 t^{2H}}\right),$$

where $m_t = E(X_t)$.

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